BOULOT, G., ROJAS, C., BENTLEY, G. A., POLJAK, R. J., BARBIER, E., LE GUERN, C & CAZENAVE, P. A. (1987). *J. Mol. Biol.* **194**, 577-578.
BRÜNGER, A. T. (1990). *Acta Cryst.* A**46**, 46-57.
BRÜNGER, A. T., LEAHY, D. J., HYNES, T. R. & FOX, R. O. (1991). *J. Mol. Biol.* **221**, 239-256.
COLMAN, P. M., DEISENHOFER, J., HUBER, R. & PALM, W. (1976). *J. Mol. Biol.* **100**, 257-282.
COLMAN, P. M., FEHLHAMMER, H. & BARTELS, K. (1976). In *Crystallographic Computing Techniques*, edited by F. R. AHMED, K. HUML & B. SEDLÁČEK, pp. 248-258. Copenhagen: Munksgaard.
CROWTHER, R. A. (1972). In *The Molecular Replacement Method*, edited by M. G. ROSSMANN, pp. 173-178. New York: Gordon & Breach.
CROWTHER, R. A. & BLOW, D. M. (1967). *Acta Cryst.* **23**, 544-548.
CYGLER, M. & ANDERSON, W. F. (1988*a*). *Acta Cryst.* A**44**, 38-45.
CYGLER, M. & ANDERSON, W. F. (1988*b*). *Acta Cryst.* A**44**, 300-308.
CYGLER, M. & DESROCHERS, M. (1989). *Acta Cryst.* A**45**, 563-572.
DOESBURG, H. M. & BEURSKENS, P. T. (1983). *Acta Cryst.* A**39**, 368-376.
DRIESSEN, H. P. C., BAX, B., SLINGSBY, C., LINDLEY, P. F., MAHADEVAN, D., MOSS, D. S. & TICKLE, I. J. (1991). *Acta Cryst.* B**47**, 987-997.
FISCHMANN, T. O., BENTLEY, G. A., BHAT, T. N., BOULOT, G., MARIUZZA, R. A., PHILLIPS, S. E. V., TELLO, D. & POLJAK, R. J. (1991). *J. Biol. Chem.* **266**, 12915-12920.
FUJINAGA, M. & READ, R. J., (1987). *J. Appl. Cryst.* **20**, 517-521.
HARADA, Y., LIFCHITZ, A., BERTHOU, J. & JOLLES, P. (1981). *Acta Cryst.* A**37**, 398-406.
HUBER, R. (1965). *Acta Cryst.* **19**, 353-356.
*International Tables for X-ray Crystallography* (1974). Vol. I. Birmingham: Kynoch Press.
KARTHA, G. & PARTHASARATHY, R. (1965). *Acta Cryst.* **18**, 749-753.
LASCOMBE, M. B., ALZARI, M., BOULOT, G., SALUDJIAN, P., TOUGARD, P., BEREK, C., HABA, S., ROSEN, E. M., NICONHOF, A. & POLJAK, R. J. (1989). *Proc. Natl Acad. Sci. USA*, **86**, 607-611.
NAVAZA, J. (1987). *Acta Cryst.* A**43**, 645-653.
READ, R. J. (1986). *Acta Cryst.* A**42**, 140-149.
READ, R. J. & SHIERBEEK, A. J. (1988). *J. Appl. Cryst.* **21**, 490-495.
ROSSMANN, M. G. (1960). *Acta Cryst.* **13**, 221-226.
ROSSMANN, M. G. & BLOW, D. M. (1962). *Acta Cryst.* **15**, 24-31.
SCHIERBEEK, A. J., SWARTE, M. B. A., DIJKSTRA, B. W., VRIEND, G., READ, R. J., HOL, W. G. J. & DRENTH, J. (1989). *J. Mol. Biol.* **206**, 365-379.
SERC Daresbury Laboratory (1986). *CCP4. A Suite of Programs for Protein Crystallography.* SERC Daresbury Laboratory, Warrington, England.
TAYLOR, C. A. & MORLEY, K. A. (1959). *Acta Cryst.* **12**, 101-105.
TICKLE, I. J. (1985). In *Molecular Replacement. Proceedings of the Daresbury Study Weekend*, 15-16 *February* 1985, edited by P. A. MACHIN, pp. 22-26.
YEATES, T. O. & RINI, J. M. (1990). *Acta Cryst.* A**46**, 352-359.

*Acta Cryst.* (1992). A**48**, 322-328

# $(p2, 2^l)$-Symmetry Three-Dimensional Space Groups $G_3^{l,p2}$

BY S. V. JABLAN

*The Mathematical Institute, Knez Mihajlova* 35, *Belgrade, Yugoslavia*

(*Received* 29 *April* 1991; *accepted* 15 *November* 1991)

## Abstract

By use of the antisymmetric characteristic method, $(p2, 2^l)$-symmetry three-dimensional space groups $G_3^{l,p2}$ ($p = 3, 4, 6$) are derived.

## Introduction

Crystallographic $(p2)$-symmetry three-dimensional space groups $G_3^{p2}$ ($p = 3, 4, 6$) were derived by Karpova (1980*a*, 1980*b*), Chubarova (1983) and Zamorzaev, Karpova, Lungu & Palistrant (1986). From 73 symmorphic space groups $G_3$ were derived 1025 junior $G_3^{p2}$ ($96\ G_3^{32} + 438\ G_3^{42} + 491\ G_3^{62}$), from 54 hemisymmorphic $G_3$, 945 junior $G_3^{p2}$ ($75\ G_3^{32} + 444\ G_3^{42} + 426\ G_3^{62}$) were derived and from 103 asymmorphic $G_3$, 1650 junior $G_3^{p2}$ ($138\ G_3^{32} + 785\ G_3^{42} + 727\ G_3^{62}$) were derived. Thus the category $G_3^{p2}$ ($p = 3, 4, 6$) consists of 3620 junior groups ($309\ G_3^{32} + 1667\ G_3^{42}$ $1644\ G_3^{62}$). By the use of the generalized antisymmetric characteristic method (AC method) introduced by the author (Jablan, 1990, 1991), some of these results will be corrected and all the crystallographic space groups of colour multiple antisymmetry $G_3^{l,p2}$ ($p = 3$, 4, 6 ) will be derived.

0108-7673/92/030322-07$03.00

## 1. Some general remarks on $(p2)$ and $(p2, 2^l)$ symmetry

$(p2)$ symmetry is a particular case of general $P$ symmetry with $P = D_p$, where $D_p$ is the irregular dihedral permutation group, generated by the permutations $e_1 = 1, \ldots, p$ and $e_2 = (2\ p), \ldots, ([(p+1)/2]\ p - [(p+1)/2] + 2)$ ($p \geq 3$), satisfying the relations

$$e_1^p = e_2^2 = (e_1 e_2)^2 = E.$$

For $p = 4q + 2$ ($q \in N$), the group $D_p$ is reducible, so the relationship

$$D_{4q+2} \simeq \{e_1^2, e_2\} \times \{e_1^{2q+1}\} = D_{2q+1} \times C_2$$

holds, where $\{e_1^2, e_2\}$ and $\{e_1^{2q+1}\}$ denote, respectively, the groups $D_{2q+1}$ and $C_2$ generated by $e_1^2$, $e_2$ and $e_1^{2q+1}$ (Zamorzaev, Galyarskij & Palistrant, 1978; Zamorzaev, Karpova, Lungu & Palistrant, 1986).

By introducing $l$ anti-identity transformations $e_3, \ldots, e_{l+2}$ ($l \in N$) (Zamorzaev, 1976; Zamorzaev & Palistrant, 1980) commuting between themselves and with $e_1$ and $e_2$, we have $(p2, 2^l)$ symmetry, which is the group $P = D_p \times C_2^l$. At $p = 4q+2$, because of the reducibility of the group $D_{4q+2}$, we have the relationship $D_{2q+1} \times C_2^l = D_{4q+2} \times C_2^{l-1}$, indicating the equivalence of $[(2q+1)2, 2^l]$ and $[(4q+2)2, 2^{l-1}]$ symmetry.

*Definition* 1: ($a$) A $(p2, 2^l)$ symmetry group is called the $M^m$-type $(p2, 2^l)$ symmetry group if it is an $M^m$-type group regarded as an $l$-multiple antisymmetry group.

($b$) A $(p2, 2^l)$ symmetry group $G^{l,p2}$ has complete $(p2, 2^l)$ symmetry if, for every $i$ ($i = 1, 2, \ldots, l+2$), the $e_i$ transformation can be obtained in the group $G^{l,p2}$ as an independent $(p2, 2^l)$ symmetry transformation. Otherwise, the group $G^{l,p2}$ has incomplete $(p2, 2^l)$ symmetry.

($c$) A complete $(p2, 2^l)$ symmetry group $G^{l,p2}$ is a junior $(p2, 2^l)$ symmetry group if all the relations given in the presentation of its generating symmetry group $G$ remain satisfied after replacing the generators of the group $G$ by the corresponding $(p2, 2^l)$-symmetry-group generators.

In this work only the junior $M^m$-type $(p2, 2^l)$ symmetry groups will be considered. Because of the aforementioned relation between $[(2q+1)2, 2^l]$ and $[(4q+2)2, 2^{l-1}]$ symmetry groups, to derive all crystallographic $(p2, 2^l)$ symmetry groups ($p = 3, 4, 6$), (32) and (42) symmetry groups will be sufficient. The derivation will be realized by use of the generalized AC method.

*Definition* 2: Let all the products of $(p2)$ symmetry generators of a group $G^{p2}$, within which each generator participates at most once, be formed and then subsets of transformations equivalent with regard to $(p2)$ symmetry be separated. The resulting system is called the antisymmetric characteristic of the group $G^{p2}$ and denoted by $AC(G^{p2})$ (Jablan, 1990, 1991).

In every transition from the antisymmetric characteristic $AC(G)$ of a generating group $G$ to the antisymmetric characteristic $AC(G^{p2})$ of a $(p2)$ symmetry group $G^{p2}$ derived from it, at $p = 2q+1$ every $e_1$ transformation has no influence on the existential conditions for the junior $M^m$ type $(p2, 2^l)$ symmetry groups (definition 1) and does not produce any change of the $AC(G)$; its possible change can be produced only by $e_2$ transformation. Hence, we can conclude that the derivation of $[(2q+1)2, 2^l]$ symmetry groups from a $[(2q+1), 2]$ symmetry group coincides with the derivation of $(l+1)$-multiple antisymmetry $M^m$-type groups ($2 \le m \le l+1$) from its corresponding $e_2$-antisymmetry group. Owing to the equivalence of $[(2q+1)2, 2^l]$ and $[(4q+2)2, 2^{l-1}]$ symmetry, the only non-trivial problem is the derivation of $(p2, 2^l)$ symmetry groups at $p = 0 \pmod 4$ (Jablan, 1991).

At $p = 0 \pmod 2$ the problem of differentiating between complete and incomplete junior $M^m$-type $(p2, 2^l)$ symmetry groups can be simply solved by the use of the homomorphism of the subgroup $C_p = \{e_1\}$ of the group $D_p$ to the group $C_2$:

$$e_1^{2k-1} \to e_1, \qquad e_1^{2k} \to E \; [1 \le k \le (p+1)/2].$$

## 2. $(p2, 2^l)$-symmetry three-dimensional space groups $G_3^{l,p2}$ ($p = 3, 4, 6$)

The application of the theoretical assumptions given above will be illustrated by examples of junior $(p2, 2^l)$-symmetry three-dimensional space groups of $M^m$ type ($p = 3, 4, 6$) derived in the family with the common generating symmetry group $G = 7s$ $(P2/m)$, $\{a, b, c\}(2:m)$ with $AC$ $\{m, cm\}\{2, 2a, 2b, 2ab\}$ belonging to the $AC$ equivalency class VII [Jablan, 1987, Table 1]. At $p = 3$ we have two junior (32) symmetry groups (Karpova, 1980$a$):

(1) $\{a, b, c^{(3)}\}(2:m^{2)})$;
(2) $\{a^{(3}, b, c\}(2^{2)}:m)$.

Because of the $e_2$ transformation $m^{2)}$, the $AC$ of the first group is of the form $\{e_2, e_2\}\{E, E, E, E\}$ and the $AC$ of the second is of the form $\{E, E\}\{e_2, e_2, e_2, e_2\}$. Hence, for both of them, $N_1 = 7$, $N_2 = 64$, $N_3 = 700$, $N_4 = 6720$ (Jablan, 1987). So we have the following junior (32, 2) symmetry groups:

$$\{\underline{a}, b, c^{(3)}\}(2:m^{2)}), \qquad \{a, b, \underline{c}^{(3)}\}(2:m^{2)}),$$
$$\{a, b, c^{(3)}\}(\underline{2}:m^{2)}), \qquad \{\underline{a}, b, \underline{c}^{(3)}\}(2:m^{2)}),$$
$$\{\underline{a}, b, c^{(3)}\}(2:\underline{m}^{2)}), \qquad \{a, b, \underline{c}^{(3)}\}(\underline{2}:m^{2)}),$$
$$\{a, b, c^{(3)}\}(\underline{2}:\underline{m}^{2)}), \qquad \{\underline{a}^{(3}, b, c\}(2^{2)}:m),$$
$$\{a^{(3}, b, \underline{c}\}(2^{2)}:m), \qquad \{a^{(3}, b, c\}(2^{2)}:\underline{m}),$$
$$\{\underline{a}^{(3}, b, \underline{c}\}(2^{2)}:m), \qquad \{\underline{a}^{(3}, b, c\}(2^{2)}:\underline{m}),$$
$$\{a^{(3}, b, \underline{c}\}(\underline{2}^{2)}:m) \qquad \{a^{(3}, b, c\}(\underline{2}^{2)}:\underline{m}),$$

where the antisymmetries are underlined.

Because of the equivalence between (32, 2) symmetry and (62) symmetry, to the 14 junior (32, 2) symmetry groups obtained correspond the 14 (62) symmetry junior groups:

$$\{a^{(2}, b, c^{(3)}\}(2:m^{2)}), \qquad \{a, b, c^{(6)}\}(2:m^{2)}),$$
$$\{a, b, c^{(3)}\}(2^{(2}:m^{2)}), \qquad \{a^{(2}, b, c^{(6)}\}(2:m^{2)}),$$
$$\{a^{(2}, b, c^{(3)}\}(2:m^{22)}), \qquad \{a, b, c^{(6)}\}(2^{(2}:m^{2)}),$$
$$\{a, b, c^{(3)}\}(2^{(2}:m^{22)}), \qquad \{a^{(6}, b, c\}(2^{2)}:m),$$
$$\{a^{(3}, b, c^{(2}\}(2^{2)}:m), \qquad \{a^{(3}, b, c\}(2^{2)}:m^{(2)}),$$
$$\{a^{(6}, b, c^{(2}\}(2^{2)}:m), \qquad \{a^{(6}, b, c\}(2^{2)}:m^{(2)}),$$
$$\{a^{(3}, b, c^{(2}\}(2^{22)}:m) \qquad \{a^{(3}, b, c\}(2^{22)}:m^{(2)}).$$

Since $(32, 2^l)$ and $(62, 2^{l-1})$ symmetry are equivalent, we may conclude that from these 14 (62) symmetry groups, $N_1 = 128$ junior (62, 2) symmetry groups of the $M^1$ type, $N_2 = 1400$ junior $(62, 2^2)$ symmetry groups of the $M^2$ type and $N_3 = 13\,420$ junior $(62, 2^3)$ symmetry groups of the $M^3$ type will be derived.

At $p = 0 \pmod 2$, the form of $AC(G^{p2})$ is obtained using the homomorphism mentioned in § 1. By treating six (42) symmetry groups in this way, we have the following results: the first three of them, $\{a, b, c^{(4)}\}\cdot(2: m^{2)})$, $\{a, b, c^{(4}\}(2^{(2}: m^{2)})$ and $\{a^{(2}, b, c^{(4}\}(2: m^{2)})$ possess $AC$ of the form $\{e_2, e_1e_2\}\{E, E, E, E\}$ and, for each of them, $N_1 = 8$, $N_2 = 64$ and $N_3 = 448$. For example, from the first group $\{a, b, c^{(4}\}(2: m^{2)})$, the eight junior (42, 2) symmetry groups of $M^1$ type:

$\{\underline{a}, b, c^{(4}\}(2: m^{2)})$, $\{a, b, c^{(4}\}(\underline{2}: m^{2)})$,

$\{\underline{a}, b, \underline{c}^{(4}\}(2: m^{2)})$, $\{\underline{a}, b, c^{(4}\}(2: \underline{m}^{2)})$,

$\{a, b, \underline{c}^{(4}\}(\underline{2}: m^{2)})$, $\{a, b, c^{(4}\}(\underline{2}: \underline{m}^{2)})$,

$\{\underline{a}, b, \underline{c}^{(4}\}(2: \underline{m}^{2)})$ $\{a, b, \underline{c}^{(4}\}(\underline{2}: \underline{m}^{2)})$

will be obtained. The three other (42) symmetry groups,

$\{a^{(4}, b, c\}(2^{2)}: m)$, $\{a^{(4}, b, c\}(2^{2)}: m^{(2})$

$\{a^{(4}, b, c^{(2}\}(2^{2)}: m)$,

have $AC$ of the form $\{E, E\}\{e_2, e_2, e_1e_2, e_1e_2\}$, so, for each of them, $N_1 = 11$, $N_2 = 132$ and $N_3 = 1344$ (Jablan, 1987). For example, from the first group, $\{a^{(4}, b, c\}(2^{2)}: m)$, the 11 junior (42, 2) symmetry groups of $M^1$ type:

$\{a^{(4}, \underline{b}, c\}(2^{2)}: m)$, $\{a^{(4}, b, \underline{c}\}(2^{2)}: m)$,

$\{a^{(4}, b, c\}(2^{2)}: \underline{m})$, $\{\underline{a}^{(4}, b, \underline{c}\}(2^{2)}: m)$,

$\{\underline{a}^{(4}, b, c\}(2^{2)}: \underline{m})$, $\{a^{(4}, \underline{b}, \underline{c}\}(2^{2)}: m)$,

$\{a^{(4}, \underline{b}, c\}(2^{2)}: \underline{m})$, $\{a^{(4}, b, \underline{c}\}(\underline{2}^{2)}: m)$,

$\{a^{(4}, b, c\}(\underline{2}^2: \underline{m})$, $\{\underline{a}^{(4}, b, \underline{c}\}(\underline{2}^2: m)$

$\{\underline{a}^{(4}, b, c\}(\underline{2}^2: \underline{m})$

will be obtained.

In the same manner, the partial catalogue of all junior $(p2, 2^l)$-symmetry three-dimensional space groups of $M^m$-type, $G_3^{l,p2}$ $(p = 3, 4, 6)$, is produced, making possible their complete cataloguing (Jablan, 1987). The final results corresponding respectively to symmorphic, hemisymmorphic and asymmorphic $(p2, 2^l)$ symmetry groups are summarized in Tables 1, 2 and 3. The connection between the international symbols (*International Tables for Crystallography*, 1987) and the symbols of symmorphic, hemisymmorphic and asymmorphic groups introduced by Fedorov can be established using the data from the monographs by Koptsik (1966) and Zamorzaev (1976).

Table 1. *Catalogue of junior $M^m$-type $(p2, 2^l)$-symmetry symmorphic three-dimensional space groups $G_3^{l,p2}$*

| | (32) | (42) | (62) | | (32) | (42) | (62) | | (32) | (42) | (62) |
|---|---|---|---|---|---|---|---|---|---|---|---|
| 2s | 1 | 1 | 1 | 24s | | 16 | | 48s | 3 | 2 | 12 |
| 3s | 1 | 2 | 4 | 25s | | 14 | | 49s | 1 | | 2 |
| 4s | 1 | 5 | 3 | 26s | 1 | 6 | 3 | 50s | 2 | | 12 |
| 5s | 1 | 2 | 4 | 27s | 1 | 3 | 1 | 51s | 3 | 1 | 3 |
| 6s | 1 | 6 | 4 | 28s | 1 | 9 | 7 | 52s | 1 | 1 | 1 |
| 7s | 2 | 6 | 14 | 29s | 1 | 14 | 6 | 53s | 2 | 2 | 8 |
| 8s | 2 | 14 | 16 | 30s | 1 | 20 | 7 | 54s | 5 | 2 | 20 |
| 9s | 1 | 4 | 8 | 31s | 1 | 20 | 6 | 55s | 3 | 2 | 12 |
| 10s | 2 | 14 | 22 | 32s | 1 | 17 | 8 | 56s | 4 | 2 | 16 |
| 11s | 1 | 6 | 4 | 33s | 1 | 19 | 7 | 57s | 2 | 2 | 8 |
| 12s | 1 | 6 | 5 | 34s | 1 | 14 | 6 | 58s | 3 | 4 | 30 |
| 13s | 1 | 6 | 16 | 35s | 1 | 12 | 4 | 63s | | 2 | |
| 14s | 1 | 12 | 14 | 36s | 1 | 30 | 16 | 65s | 1 | | 2 |
| 15s | 2 | 16 | 20 | 37s | 1 | 32 | 14 | 66s | 1 | | 2 |
| 16s | 1 | 8 | 6 | 40s | 2 | | 4 | 67s | 1 | | |
| 17s | 1 | 8 | 14 | 41s | 1 | | 2 | 68s | 1 | | 1 |
| 18s | 1 | 6 | 16 | 42s | 1 | | 2 | 69s | 1 | 2 | 2 |
| 19s | 2 | 24 | 38 | 43s | 1 | 1 | 1 | 70s | 1 | | |
| 20s | 1 | 12 | 10 | 44s | 4 | 1 | 4 | 71s | 1 | | 4 |
| 21s | 1 | 12 | 18 | 45s | 6 | 1 | 6 | 72s | 1 | 4 | 6 |
| 22s | | 4 | | 46s | 4 | 1 | 4 | 73s | 1 | | 2 |
| 23s | | 6 | | 47s | 2 | 2 | 8 | | | | |

| | (32, 2) | (42, 2) | (62, 2) | $(32, 2^2)$ | $(42, 2^2)$ | $(62, 2^2)$ |
|---|---|---|---|---|---|---|
| 2s | 1 | 1 | 1 | 1 | 1 | 1 |
| 3s | 4 | 12 | 16 | 16 | 48 | 56 |
| 4s | 3 | 8 | 6 | 6 | | |
| 5s | 4 | 8 | 22 | 22 | 32 | 112 |
| 6s | 4 | 12 | 12 | 12 | | |
| 7s | 14 | 57 | 128 | 128 | 588 | 1400 |
| 8s | 16 | 96 | 120 | 120 | 480 | 672 |
| 9s | 8 | 32 | 96 | 96 | 432 | 1516 |
| 10s | 22 | 120 | 228 | 228 | 768 | 1680 |
| 11s | 4 | 12 | 12 | 12 | | |
| 12s | 5 | 36 | 42 | 42 | 288 | 336 |
| 13s | 16 | 120 | 300 | 300 | 2304 | 5712 |
| 14s | 14 | 128 | 168 | 168 | 960 | 1344 |
| 15s | 20 | 144 | 192 | 192 | 960 | 2688 |
| 16s | 6 | 24 | 24 | 24 | | |
| 17s | 14 | 96 | 168 | 168 | 768 | 1344 |
| 18s | 16 | 120 | 450 | 450 | 3744 | 17220 |
| 19s | 38 | 444 | 804 | 804 | 8208 | 16464 |
| 20s | 10 | 88 | 96 | 96 | 512 | 672 |
| 21s | 18 | 336 | 432 | 432 | 8064 | 10080 |
| 22s | | 16 | | | | |
| 24s | | 160 | | | 1152 | |
| 25s | | 56 | | | | |
| 26s | 3 | 18 | 6 | 6 | | |
| 27s | 1 | | | | | |
| 28s | 7 | 72 | 54 | 54 | 432 | 336 |
| 29s | 6 | 56 | 24 | 24 | | |
| 30s | 7 | 136 | 54 | 54 | 768 | 336 |
| 31s | 6 | 80 | 24 | 24 | | |
| 32s | 8 | 156 | 60 | 60 | 1056 | 336 |
| 33s | 7 | 104 | 54 | 54 | 528 | 336 |
| 34s | 6 | 56 | 24 | 24 | | |
| 35s | 4 | 40 | 12 | 12 | | |
| 36s | 16 | 528 | 300 | 300 | 9504 | 5712 |
| 37s | 14 | 384 | 168 | 168 | 3072 | 1344 |
| 40s | 4 | | | | | |
| 41s | 2 | | | | | |
| 42s | 2 | | | | | |
| 43s | 1 | | | | | |
| 44s | 4 | | | | | |
| 45s | 6 | | | | | |
| 46s | 4 | | | | | |
| 47s | 8 | 8 | 24 | 24 | | |
| 48s | 12 | 8 | 36 | 36 | | |
| 49s | 2 | | | | | |
| 50s | 12 | | 48 | 48 | | |
| 51s | 3 | | | | | |
| 52s | 1 | | | | | |

Table 1 (*cont.*)

| | (32, 2) | (42, 2) | (62, 2) | (32, $2^2$) | (42, $2^2$) | (62, $2^2$) |
|---|---|---|---|---|---|---|
| 53*s* | 8 | 8 | 24 | 24 | | |
| 54*s* | 20 | 8 | 60 | 60 | | |
| 55*s* | 12 | 8 | 36 | 36 | | |
| 56*s* | 16 | 8 | 48 | 48 | | |
| 57*s* | 8 | 8 | 24 | 24 | | |
| 58*s* | 30 | 48 | 288 | 288 | 384 | 2016 |
| 65*s* | 2 | | | | | |
| 66*s* | 2 | | | | | |
| 68*s* | 1 | | | | | |
| 69*s* | 2 | | | | | |
| 71*s* | 4 | | | | | |
| 72*s* | 6 | 16 | 24 | 24 | | |
| 73*s* | 2 | | | | | |

| | (32, $2^3$) | (42, $2^3$) | (62, $2^3$) | (32, $2^4$) | (42, $2^4$) | (62, $2^4$) | (32, $2^5$) |
|---|---|---|---|---|---|---|---|
| 2*s* | 1 | | | | | | |
| 3*s* | 56 | | | | | | |
| 5*s* | 112 | | | | | | |
| 7*s* | 1400 | 5376 | 13440 | 13440 | | | |
| 8*s* | 672 | | | | | | |
| 9*s* | 1516 | 5376 | 20160 | 20160 | | | |
| 10*s* | 1680 | | | | | | |
| 12*s* | 336 | | | | | | |
| 13*s* | 5712 | 32256 | 80640 | 80640 | | | |
| 14*s* | 1344 | | | | | | |
| 15*s* | 2688 | | | | | | |
| 17*s* | 1344 | | | | | | |
| 18*s* | 17220 | 137088 | 685440 | 685440 | 3870720 | 19998720 | 19998720 |
| 19*s* | 16464 | 112896 | 241920 | 241920 | | | |
| 20*s* | 672 | | | | | | |
| 21*s* | 10080 | 129024 | 161280 | 161280 | | | |
| 28*s* | 336 | | | | | | |
| 30*s* | 336 | | | | | | |
| 32*s* | 336 | | | | | | |
| 33*s* | 336 | | | | | | |
| 36*s* | 5712 | 129024 | 80640 | 80640 | | | |
| 37*s* | 1344 | | | | | | |
| 58*s* | 2016 | | | | | | |

Table 2. *Catalogue of junior $M^m$-type $(p2, 2^l)$-symmetry hemisymmorphic three-dimensional space groups $G_3^{l,p2}$*

| | (32) | (42) | (62) | | (32) | (42) | (62) | | (32) | (42) | (62) |
|---|---|---|---|---|---|---|---|---|---|---|---|
| 1*h* | 1 | 2 | 3 | 19*h* | 1 | 6 | 5 | 37*h* | 1 | 20 | 6 |
| 2*h* | 1 | 4 | 1 | 20*h* | 2 | 22 | 24 | 38*h* | 1 | 28 | 14 |
| 3*h* | 2 | 9 | 12 | 21*h* | 2 | 22 | 36 | 39*h* | 2 | | |
| 4*h* | 2 | 10 | 8 | 22*h* | 2 | 24 | 22 | 40*h* | 1 | | |
| 5*h* | 1 | 3 | 7 | 23*h* | 2 | 18 | 24 | 41*h* | 1 | | |
| 6*h* | 2 | 10 | 20 | 24*h* | 1 | 4 | 4 | 42*h* | 2 | 2 | 4 |
| 7*h* | 2 | 11 | 8 | 25*h* | | 8 | | 43*h* | 3 | 2 | 6 |
| 8*h* | 1 | 4 | 3 | 26*h* | | 8 | | 44*h* | 2 | | 4 |
| 9*h* | 1 | 4 | 6 | 27*h* | | 6 | | 45*h* | 3 | 2 | 6 |
| 10*h* | 1 | 8 | 6 | 28*h* | | 10 | | 46*h* | 4 | 2 | 8 |
| 11*h* | 2 | 12 | 20 | 29*h* | 1 | 11 | 4 | 47*h* | 2 | 2 | 4 |
| 12*h* | 2 | 16 | 12 | 30*h* | 1 | 14 | 4 | 48*h* | 3 | 4 | 18 |
| 13*h* | 2 | 10 | 12 | 31*h* | 1 | 12 | 4 | 51*h* | 1 | | |
| 14*h* | 2 | 6 | 12 | 32*h* | 1 | 8 | 4 | 52*h* | 1 | | |
| 15*h* | 1 | 6 | 6 | 33*h* | 1 | 8 | 3 | 53*h* | 1 | | 2 |
| 16*h* | 1 | 2 | 2 | 34*h* | 1 | 10 | 4 | 54*h* | 1 | | 2 |
| 17*h* | 2 | 14 | 26 | 35*h* | 1 | 22 | 10 | | | | |
| 18*h* | 2 | 12 | 17 | 36*h* | 1 | 26 | 10 | | | | |

Table 2 (*cont.*)

| | (32, 2) | (42, 2) | (62, 2) | (32, $2^2$) | (42, $2^2$) | (62, $2^2$) |
|---|---|---|---|---|---|---|
| 1*h* | 3 | 8 | 6 | 6 | | |
| 2*h* | 1 | | | | | |
| 3*h* | 12 | 58 | 72 | 72 | 264 | 336 |
| 4*h* | 8 | 36 | 24 | 24 | | |
| 5*h* | 7 | 24 | 54 | 54 | 144 | 336 |
| 6*h* | 20 | 96 | 192 | 192 | 672 | 1344 |
| 7*h* | 8 | 28 | 24 | 24 | | |
| 8*h* | 3 | 4 | 6 | 6 | | |
| 9*h* | 6 | 16 | 24 | 24 | | |
| 10*h* | 6 | 24 | 24 | 24 | | |
| 11*h* | 20 | 112 | 192 | 192 | 768 | 1344 |
| 12*h* | 12 | 64 | 48 | 48 | | |
| 13*h* | 12 | 40 | 48 | 48 | | |
| 14*h* | 12 | 24 | 48 | 48 | | |
| 15*h* | 6 | 20 | 24 | 24 | | |
| 16*h* | 2 | | | | | |
| 17*h* | 26 | 184 | 456 | 456 | 3168 | 8568 |
| 18*h* | 17 | 90 | 150 | 150 | 528 | 1008 |
| 19*h* | 5 | 18 | 24 | 24 | 36 | 84 |
| 20*h* | 24 | 208 | 264 | 264 | 1440 | 2016 |
| 21*h* | 36 | 416 | 768 | 768 | 8160 | 16128 |
| 22*h* | 22 | 240 | 252 | 252 | 1920 | 2016 |
| 23*h* | 24 | 176 | 264 | 264 | 1248 | 2016 |
| 24*h* | 4 | 8 | 12 | 12 | | |
| 25*h* | | 24 | | | | |
| 26*h* | | 32 | | | | |
| 28*h* | | 40 | | | | |
| 29*h* | 4 | 28 | 12 | 12 | | |
| 30*h* | 4 | 40 | 12 | 12 | | |
| 31*h* | 4 | 32 | 12 | 12 | | |
| 32*h* | 4 | 20 | 12 | 12 | | |
| 33*h* | 3 | 8 | 6 | 6 | | |
| 34*h* | 4 | 32 | 12 | 12 | | |
| 35*h* | 10 | 208 | 96 | 96 | 1440 | 672 |
| 36*h* | 10 | 232 | 96 | 96 | 1536 | 672 |
| 37*h* | 6 | 80 | 24 | 24 | | |
| 38*h* | 14 | 336 | 168 | 168 | 2688 | 1344 |
| 42*h* | 4 | | | | | |
| 43*h* | 6 | | | | | |
| 44*h* | 4 | | | | | |
| 45*h* | 6 | | | | | |
| 46*h* | 8 | | | | | |
| 47*h* | 4 | | | | | |
| 48*h* | 18 | 16 | 72 | 72 | | |
| 53*h* | 2 | | | | | |
| 54*h* | 2 | | | | | |

| | (32, $2^3$) | (42, $2^3$) | (62, $2^3$) | (32, $2^4$) |
|---|---|---|---|---|
| 3*h* | 336 | | | |
| 5*h* | 336 | | | |
| 6*h* | 1344 | | | |
| 11*h* | 1344 | | | |
| 17*h* | 8568 | 43008 | 120960 | 120960 |
| 18*h* | 1008 | | | |
| 19*h* | 84 | | | |
| 20*h* | 2016 | | | |
| 21*h* | 16128 | 118272 | 241920 | 241920 |
| 22*h* | 2016 | | | |
| 23*h* | 2016 | | | |
| 35*h* | 672 | | | |
| 36*h* | 672 | | | |
| 38*h* | 1344 | | | |

From the results of the derivation of the junior (32, 2)-symmetry symmorphic groups, we can first conclude that, because of the equivalence between (32, 2) and (62) symmetry, there exist 496 (not 491) (Karpova 1980*a*; Chubarova, 1983; Zamorzaev, Karpova, Lungu & Palistrant, 1986) junior (32, 2)-symmetry symmorphic three-dimensional space groups and, consequently, the same number of symmorphic (62) symmetry groups. This difference from the works mentioned implies that corrections are necessary for the (62) symmetry groups in the families with the generating groups 10*s* (*C*222) and 32*s* ($P\bar{4}2m$). For the first family, the correct number of (62) symmetry groups is 22 (not 18) and for the second it is 8 (not 7) (Karpova, 1980*a*).

Table 3. *Catalogue of junior $M^m$-type $(p2, 2^l)$-symmetry asymmorphic three-dimensional space groups $G_3^{(l,p2)}$*

| | (32) | (42) | (62) |
|---|---|---|---|
| 1*a* | 1 | 1 | 1 |
| 2*a* | 2 | 6 | 8 |
| 3*a* | 2 | 7 | 6 |
| 4*a* | 2 | 9 | 11 |
| 5*a* | 2 | 10 | 10 |
| 6*a* | 1 | 4 | 4 |
| 7*a* | 2 | 6 | 9 |
| 8*a* | 1 | 2 | 1 |
| 9*a* | 2 | 6 | 14 |
| 10*a* | 2 | 10 | 8 |
| 11*a* | 2 | 6 | 8 |
| 12*a* | 2 | 6 | 4 |
| 13*a* | 2 | 10 | 12 |
| 14*a* | 3 | 24 | 48 |
| 15*a* | 3 | 22 | 30 |
| 16*a* | 3 | 16 | 30 |
| 17*a* | 3 | 18 | 18 |
| 18*a* | 3 | 28 | 42 |
| 19*a* | 3 | 22 | 42 |
| 20*a* | 2 | 10 | 22 |
| 21*a* | 1 | 4 | 10 |
| 22*a* | 2 | 9 | 17 |
| 23*a* | 3 | 18 | 30 |
| 24*a* | 2 | 15 | 17 |
| 25*a* | 2 | 12 | 10 |
| 26*a* | 3 | 14 | 18 |
| 27*a* | 2 | 10 | 10 |
| 28*a* | 3 | 14 | 18 |
| 29*a* | 1 | 4 | 6 |
| 30*a* | | 2 | |
| 31*a* | | 2 | |
| 32*a* | | 4 | |
| 33*a* | | 4 | |

| | (32) | (42) | (62) |
|---|---|---|---|
| 34*a* | | 4 | |
| 35*a* | | 4 | |
| 36*a* | | 14 | |
| 37*a* | | 12 | |
| 38*a* | | 8 | |
| 39*a* | | 6 | |
| 40*a* | 1 | 4 | 6 |
| 41*a* | 1 | 14 | 4 |
| 42*a* | 1 | 8 | 3 |
| 43*a* | 1 | 12 | 4 |
| 44*a* | 1 | 8 | 4 |
| 45*a* | 1 | 8 | 4 |
| 46*a* | 1 | 10 | 6 |
| 47*a* | 1 | 14 | 7 |
| 48*a* | 1 | 4 | 2 |
| 49*a* | 1 | 4 | 2 |
| 50*a* | 1 | 8 | 4 |
| 51*a* | 1 | 4 | 2 |
| 52*a* | 1 | 10 | 4 |
| 53*a* | 1 | 6 | 2 |
| 54*a* | 1 | 16 | 10 |
| 55*a* | 1 | 22 | 10 |
| 56*a* | 1 | 14 | 6 |
| 57*a* | 1 | 16 | 6 |
| 58*a* | 1 | 12 | 6 |
| 59*a* | 1 | 12 | 6 |
| 60*a* | 1 | 30 | 10 |
| 61*a* | 1 | 28 | 10 |
| 62*a* | 1 | 20 | 8 |
| 63*a* | 1 | 20 | 6 |
| 64*a* | 1 | 20 | 6 |
| 65*a* | 1 | 20 | 6 |
| 66*a* | 1 | 14 | 6 |

| | (32) | (42) | (62) |
|---|---|---|---|
| 67*a* | 1 | 20 | 6 |
| 70*a* | 1 | 1 | 1 |
| 71*a* | 1 | 1 | 1 |
| 72*a* | 2 | 1 | 2 |
| 73*a* | 2 | 1 | 2 |
| 74*a* | 1 | | |
| 75*a* | 1 | | |
| 76*a* | 1 | | |
| 77*a* | 1 | | |
| 78*a* | 1 | | |
| 79*a* | 2 | | 4 |
| 80*a* | 2 | | 4 |
| 81*a* | 3 | 2 | 6 |
| 82*a* | 2 | 2 | 4 |
| 83*a* | 2 | 2 | 4 |
| 84*a* | 2 | 2 | 8 |
| 85*a* | 2 | 2 | 8 |
| 86*a* | 5 | 2 | 10 |
| 87*a* | 3 | 4 | 18 |
| 88*a* | 3 | 4 | 18 |
| 93*a* | 1 | | |
| 94*a* | 1 | | |
| 95*a* | 1 | | |
| 96*a* | 1 | | 2 |
| 97*a* | 1 | | |
| 98*a* | 1 | | 1 |
| 99*a* | 1 | | 2 |
| 100*a* | 1 | | 2 |
| 101*a* | 1 | | 2 |
| 102*a* | 1 | | 2 |
| 103*a* | 1 | | 4 |

| | (32, 2) | (42, 2) | (62, 1) | (32, $2^2$) | (42, $2^2$) | (62, $2^2$) |
|---|---|---|---|---|---|---|
| 1*a* | 1 | 1 | 1 | 1 | | |
| 2*a* | 8 | 20 | 32 | 32 | 64 | 112 |
| 3*a* | 6 | 10 | 12 | 12 | | |
| 4*a* | 11 | 54 | 81 | 81 | 288 | 504 |
| 5*a* | 10 | 36 | 36 | 36 | | |
| 6*a* | 4 | 12 | 12 | 12 | | |
| 7*a* | 9 | 18 | 30 | 30 | | |
| 8*a* | 1 | | | | | |
| 9*a* | 14 | 48 | 108 | 108 | 288 | 672 |
| 10*a* | 8 | 24 | 24 | 24 | | |
| 11*a* | 8 | 16 | 24 | 24 | | |
| 12*a* | 4 | | | | | |
| 13*a* | 12 | 40 | 48 | 48 | | |
| 14*a* | 48 | 408 | 900 | 900 | 7200 | 17136 |
| 15*a* | 30 | 192 | 288 | 288 | 1248 | 2016 |
| 16*a* | 30 | 144 | 288 | 288 | 960 | 2016 |
| 17*a* | 18 | 72 | 72 | 72 | | |
| 18*a* | 42 | 336 | 504 | 504 | 2688 | 4032 |
| 19*a* | 42 | 264 | 504 | 504 | 2112 | 4032 |
| 20*a* | 22 | 120 | 252 | 252 | 960 | 2016 |
| 21*a* | 10 | 48 | 96 | 96 | 384 | 672 |
| 22*a* | 17 | 72 | 150 | 150 | 432 | 1008 |
| 23*a* | 30 | 160 | 288 | 288 | 1056 | 2016 |
| 24*a* | 17 | 108 | 150 | 150 | 624 | 1008 |
| 25*a* | 10 | 36 | 36 | 36 | | |
| 26*a* | 18 | 56 | 72 | 72 | | |
| 27*a* | 10 | 28 | 36 | 36 | | |
| 28*a* | 18 | 56 | 72 | 72 | | |
| 29*a* | 6 | 16 | 24 | 24 | | |
| 33*a* | | 16 | | | | |
| 36*a* | | 40 | | | | |
| 37*a* | | 32 | | | | |
| 38*a* | | 16 | | | | |

Table 3 (*cont.*)

| | (32, 2) | (42, 2) | (62, 1) | (32, $2^2$) | (42, $2^2$) | (62, $2^2$) |
|---|---|---|---|---|---|---|
| 40*a* | 6 | 16 | 24 | 24 | | |
| 41*a* | 4 | 56 | 12 | 12 | | |
| 42*a* | 3 | 8 | 6 | 6 | | |
| 43*a* | 4 | 32 | 12 | 12 | | |
| 44*a* | 4 | 24 | 12 | 12 | | |
| 45*a* | 4 | 24 | 12 | 12 | | |
| 46*a* | 6 | 40 | 24 | 24 | | |
| 47*a* | 7 | 88 | 54 | 54 | 480 | 336 |
| 48*a* | 2 | | | | | |
| 49*a* | 2 | | | | | |
| 50*a* | 4 | 16 | 12 | 12 | | |
| 51*a* | 2 | | | | | |
| 52*a* | 4 | 24 | 12 | 12 | | |
| 53*a* | 2 | | | | | |
| 54*a* | 10 | 144 | 96 | 96 | 960 | 672 |
| 55*a* | 10 | 192 | 96 | 96 | 1248 | 672 |
| 56*a* | 6 | 56 | 24 | 24 | | |
| 57*a* | 6 | 64 | 24 | 24 | | |
| 58*a* | 6 | 48 | 24 | 24 | | |
| 59*a* | 6 | 48 | 24 | 24 | | |
| 60*a* | 10 | 288 | 96 | 96 | 2016 | 672 |
| 61*a* | 10 | 256 | 96 | 96 | 1728 | 672 |
| 62*a* | 8 | 120 | 60 | 60 | 480 | 336 |
| 63*a* | 6 | 80 | 24 | 24 | | |
| 64*a* | 6 | 80 | 24 | 24 | | |
| 65*a* | 6 | 80 | 24 | 24 | | |
| 66*a* | 6 | 56 | 24 | 24 | | |
| 67*a* | 6 | 80 | 24 | 24 | | |
| 70*a* | 1 | | | | | |
| 71*a* | 1 | | | | | |
| 72*a* | 2 | | | | | |
| 73*a* | 2 | | | | | |
| 79*a* | 4 | | | | | |
| 80*a* | 4 | | | | | |
| 81*a* | 6 | | | | | |
| 82*a* | 4 | | | | | |
| 83*a* | 4 | | | | | |
| 84*a* | 8 | 8 | 24 | 24 | | |
| 85*a* | 8 | 8 | 24 | 24 | | |
| 86*a* | 10 | | | | | |
| 87*a* | 18 | 16 | 72 | 72 | | |
| 88*a* | 18 | 16 | 72 | 72 | | |
| 96*a* | 2 | | | | | |
| 98*a* | 1 | | | | | |
| 99*a* | 2 | | | | | |
| 100*a* | 2 | | | | | |
| 101*a* | 2 | | | | | |
| 102*a* | 2 | | | | | |
| 103*a* | 4 | | 12 | 12 | | |

| | (32, $2^3$) | (42, $2^3$) | (62, $2^3$) | (32, $2^4$) |
|---|---|---|---|---|
| 2*a* | 112 | | | |
| 4*a* | 504 | | | |
| 9*a* | 672 | | | |
| 14*a* | 17136 | 96768 | 241920 | 241920 |
| 15*a* | 2016 | | | |
| 16*a* | 2016 | | | |
| 18*a* | 4032 | | | |
| 19*a* | 4032 | | | |
| 20*a* | 2016 | | | |
| 21*a* | 672 | | | |
| 22*a* | 1008 | | | |
| 23*a* | 2016 | | | |
| 24*a* | 1008 | | | |
| 47*a* | 336 | | | |
| 54*a* | 672 | | | |
| 55*a* | 672 | | | |
| 60*a* | 672 | | | |
| 61*a* | 672 | | | |
| 62*a* | 336 | | | |

For the junior $(p2, 2')$-symmetry symmorphic three-dimensional space groups of $M^m$ type, the numbers $N_m^{p2}$ $(p=3, 4, 6)$ are:

$$N_0^{p2}=96\ G_3^{32}+438\ G_3^{42}+496\ G_3^{62}=1030;$$

$$N_1^{p2}=496\ G_3^{1,32}+3876\ G_3^{1,42}+4709\ G_3^{1,62}=9081;$$

$$N_2^{p2}=4709\ G_3^{2,32}+45\,053\ G_3^{2,42}+71\,713\ G_3^{2,62} =121\,475;$$

$$N_3^{p2}=71\,713\ G_3^{3,32}+551\,040\ G_3^{3,42}+1\,283\,520\ G_3^{3,62} =1\,906\,273;$$

$$N_4^{p2}=1\,283\,520\ G_3^{4,32}+3\,870\,720\ G_3^{4,42} +19\,998\,720\ G_3^{4,62}=25\,152\,960;$$

$$N_5^{p2}=19\,998\,720\ G_3^{5,32}=19\,998\,720.$$

From the results of the derivation of the junior (32, 2)-symmetry hemisymmorphic groups we may first conclude that, because of the equivalence between (32, 2) and (62) symmetry, there exist 413 (not 426) (Karpova, 1980*b*; Chubarova, 1983; Zamorzaev, Karpova, Lungu & Palistrant, 1986) junior (32, 2)-symmetry hemisymmorphic three-dimensional space groups and, consequently, the same number of hemisymmorphic (62) symmetry groups. This difference from published results implies that corrections are also necessary for the (62) symmetry groups in the families with the generating groups 1*h* (*Pb*), 2*h* (*Bb*), 3*h* (*P*2/*b*), 4*h* (*B*2/*b*), 21*h* (*Cmma*), 22*h* (*Ccca*) and 53*h* (*Pn*3*n*). For these families, the correct number of (62) symmetry groups is, respectively, 3 (not 4), 1 (not 2), 12 (not 13), 8 (not 10), 36 (not 38), 22 (not 24) and 2 (not 6) (Karpova, 1980*b*).

For the junior $(p2, 2')$-symmetry hemisymmorphic three-dimensional space groups of $M^m$ type the numbers $N_m^{p2}$ $(p=3, 4, 6)$ are:

$$N_0^{p2}=75\ G_3^{32}+444\ G_3^{42}+413\ G_3^{62}=932;$$

$$N_1^{p2}=413\ G_3^{1,32}+3022\ G_3^{1,42}+3498\ G_3^{1,62}=6933;$$

$$N_2^{p2}=3498\ G_3^{2,32}+24\,012\ G_3^{2,42}+37\,884\ G_3^{2,62} =65\,394;$$

$$N_3^{p2}=37\,884\ G_3^{3,32}+161\,280\ G_3^{3,42}+362\,880\ G_3^{3,62} =562\,044;$$

$$N_4^{p2}=362\,880\ G_3^{4,32}=362\,880.$$

From the results of the derivation of the junior (32, 2)-symmetry asymmorphic groups, we may first conclude that, because of the equivalence between (32, 2) and (62) symmetry, there exist 725 (not 727) (Karpova, 1980*b*; Chubarova, 1983; Zamorzaev, Karpova, Lungu & Palistrant, 1986) junior (32, 2)-symmetry asymmorphic three-dimensional space groups and, consequently, the same number of asymmorphic (62) symmetry groups.

For the junior $(p2, 2')$-symmetry asymmorphic three-dimensional groups of $M^m$ type, the numbers $N_m^{p2}$ are:

$$N_0^{p2}=138\ G_3^{32}+785\ G_3^{42}+725\ G_3^{62}=1648;$$

$$N_1^{p2}=725\ G_3^{1,32}+4467\ G_3^{1,42}+5184\ G_3^{1,62}=10\,376;$$

$$N_2^{p2}=5184\ G_3^{2,32}+25\,216\ G_3^{2,42}+40\,600\ G_3^{2,62} =71\,000;$$

$$N_3^{p2}=40\,600\ G_3^{3,32}+96\,768\ G_3^{3,42}+241\,920\ G_3^{3,62} =379\,288;$$

$$N_4^{p2}=241\,920\ G_3^{4,32}=241\,920.$$

## 3. Concluding remarks

For the junior $(p2, 2')$-symmetry three-dimensional space groups of the $M^m$ type, the numbers $N_m(G_3^{p2})$ $(p=3, 4, 6)$ are:

$$N_0(G_3^{p2})=309\ G_3^{32}+1667\ G_3^{42}+1634\ G_3^{62}=3610;$$

$$N_1(G_3^{p2})=1634\ G_3^{1,32}+11\,365\ G_3^{1,42}+13\,391\ G_3^{1,62} =26\,390$$

$$N_2(G_3^{p2})=13\,391\ G_3^{2,32}+94\,281\ G_3^{2,42}+150\,197\ G_3^{2,62} =257\,869;$$

$$N_3(G_3^{p2})=150\,197\ G_3^{3,32}+809\,088\ G_3^{3,42} +1\,888\,320\ G_3^{3,62}=2\,847\,605;$$

$$N_4(G_3^{p2})=1\,888\,320\ G_3^{4,32}+3\,870\,720\ G_3^{4,42} +19\,998\,720\ G_3^{4,62}=25\,757\,760;$$

$$N_5(G_3^{p2})=19\,998\,720\ G_3^{5,32}=19\,998\,720.$$

The possible physical applications of the groups derived are given in the work by Koptsik (1988).

### References


CHUBAROVA, YU. S. (1983). *Derivation of Junior Space Groups of P Symmetry.* Kishinev: Shtiintsa.

*International Tables for Crystallography* (1987). Vol. A. Dordrecht: Kluwer Academic Publishers.

JABLAN, S. V. (1987). *Acta Cryst.* **A43**, 326–337.

JABLAN, S. V. (1990). *Publ. Inst. Math.* **47** (61), 39–55.

JABLAN, S. V. (1991). *Mat. Vesn.* In the press.

KARPOVA, YU. S. (1980*a*). *Derivation of Symmorphic Junior Space Groups of (p2) Symmetry.* Kishinev Univ. Press. (Deposited in VINITI, No. 855-80 Dep.)

KARPOVA, YU. S. (1980*b*). *Derivation of Hemisymmorphic and Asymmorphic Junior Space Groups of (p2) Symmetry.* Kishinev Univ. Press. (Deposited in VINITI, No. 3173-80 Dep.)

KOPTSIK, V. A. (1966). *Shubnikov groups.* Moscow Univ. Press. In Russian.

KOPTSIK, V. A. (1988). *Comput. Math. Appl.* **16**, 407–424.

ZAMORZAEV, A. M. (1976). *The Theory of Simple and Multiple Antisymmetry.* Kishinev: Shtiintsa.
ZAMORZAEV, A. M., GALYARSKIJ, E. I. & PALISTRANT, A. F. (1978). *Colored Symmetry, its Generalizations and Applications.* Kishinev: Shtiintsa.
ZAMORZAEV, A. M., KARPOVA, YU. S., LUNGU, A. P. & PALISTRANT, A. F. (1986). *P Symmetry and its Further Development.* Kishinev: Shtiintsa.
ZAMORZAEV, A. M. & PALISTRANT, A. F. (1980). *Z. Kristallogr.* **151**, 231-248.

*Acta Cryst.* (1992). A**48**, 328-335

# A Pentagonal Quasiperiodic Tiling with Fractal Acceptance Domain

BY E. ZOBETZ

*Institut für Mineralogie, Kristallographie und Strukturchemie, Technische Universität Wien, Getreidemarkt 9, A-1060 Vienna, Austria*

(*Received* 31 *May* 1991; *accepted* 22 *November* 1991)

**Abstract**

The structural properties of a pentagonal quasiperiodic tiling obtained by new deflation and matching rules are described. Two new types of vertices occur that cannot be found in generalized Penrose patterns. Tile and vertex frequencies as well as average edge valencies of vertices are enumerated. The geometry underlying the acceptance domain involves self-similar fractals. Two methods of constructing the fractally shaped acceptance domain have been found.

## 1. Introduction

Considerable progress has been made on quasiperiodic structures during the last few years. Most of the work was focused on original (Penrose, 1974, 1979; Gardner, 1977; de Bruijn, 1981; Mackay, 1982; Levine & Steinhardt, 1986; Socolar & Steinhardt, 1986; Henley, 1986; Jarić, 1986; Kumar, Sahoo & Athithan, 1986; Socolar, 1989) and generalized Penrose tilings (Duneau & Katz, 1985; Pavlovitch & Kléman, 1987; Ishihara & Yamamoto, 1988; Whittaker & Whittaker, 1988; Zobetz & Preisinger, 1990). This is not surprising since there exists a large number of tools with which to analyze them: matching and deflation rules (Penrose, 1974; Gardner, 1977; de Bruijn, 1981; Levine & Steinhardt, 1986; Socolar & Steinhardt, 1986; Socolar, 1989), grid construction methods (de Bruijn, 1981; Gähler & Rhyner, 1986; Levine & Steinhardt, 1986; Socolar & Steinhardt, 1986; Korepin, Gähler & Rhyner, 1988; Socolar, 1989) and strip-projection methods (de Bruijn, 1981; Kramer, 1982; Kramer & Neri, 1984; Duneau & Katz, 1985; Elser, 1986; Ostlund & Wright, 1986; Jarić, 1986; Kramer & Zeidler, 1989; Socolar, 1989).

Although one has to keep in mind that for any given orientational symmetry it is possible to construct an infinite number of quasiperiodic tilings that can be grouped into LI* classes (Levine & Steinhardt, 1986; Socolar & Steinhardt, 1986; Socolar, 1989), comparatively little attention has been paid to other pentagonal tilings (Gähler & Rhyner, 1986; Socolar & Steinhardt, 1986; Olami & Kléman, 1989) than original and generalized Penrose tilings. There is no reason to exclude *a priori* other tilings. Up to now we have no reason to suggest that properties of original Penrose tilings, such as matching rules, which force nonperiodicity, have direct physical significance, in the sense that the growth probability at a given site depends only on its local neighbourhood, since only growth algorithms with short-range interactions are of physical interest (Levitov, 1988; Onada, Steinhardt, DiVincenzo & Socolar, 1988, 1989; Jarić & Ronchetti, 1989; Socolar, 1990; Olami, 1991; Ingersent & Steinhardt, 1991). However, there are few proven results in this relatively new field of investigation. Hence, one is led to the conclusion that this key problem is by no means completely settled. Socolar (1989, p. 10548) stated: 'It is worth emphasizing again that the PLI† class tilings discussed in this paper, though they play a special role in the analysis of the symmetries of interest here, are not necessarily privileged candidates for modeling physical structures. It is possible that a different LI class would be required for the description of a given physical material.' Therefore tilings that have the same orientational symmetry and tile shapes as Penrose tilings, but clearly have very different local and global configurations of tiles, may also be of interest and relevance to the study of quasicrystalline materials. Baake, Schlottmann & Jarvis (1991) introduced the new equivalence concept of mutual local derivability. They prove that under special conditions a tiling can

* Two tilings are locally isomorphic and belong to the same local isomorphism (LI) class if any finite region that occurs in one also occurs in the other.

† PLI denotes the LI class of the original Penrose tilings.

0108-7673/92/030328-08$03.00 